Can One Hear the Shape of a Drum?

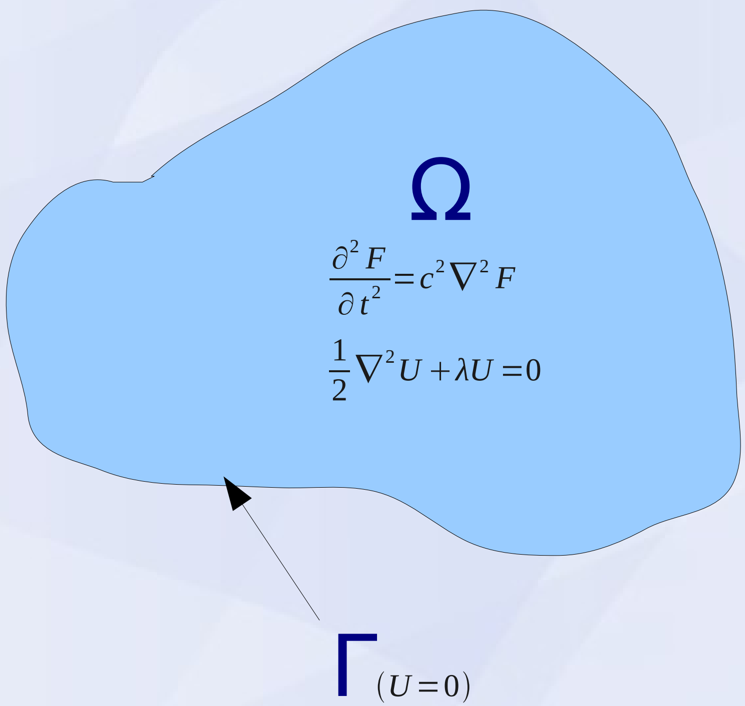
a review of a paper by Mark Kac

(see references)

Well, is it possible?

- Not known in general, still an open problem
- Some features of the shape have been shown to be deducible from eigenvalues
- This paper gives some examples of what knowledge can be gained

Exact problem statement



- Classical wave equation in 2d with c=1/2
- Assume Dirichlet boundary conditions, on area Ω with boundary Γ
- Knowing the eigenvalues λ what can we deduce about Ω

Approach

- Interested in asymptotic properties of λ for large frequencies
- Consider Dirichlet series: $\sum_{n=1}^{\infty} e^{-\lambda_n t}$
- Final result:

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4\sqrt{2\pi t}} + \frac{1}{6}(1-r) + higher\ powers\ of\ t$$

Where |Ω| is the area, L is the length of the boundary Γ and r is the number of holes in Ω

First term (approach 1)

- Consider the following object: $\frac{1}{|\Omega|}\sum_{n=1}^{\infty} e^{-\lambda_n t}$
- Apply result by Weyl: $N(\lambda) \sim \frac{|\Omega|}{2\pi}\lambda \quad , \quad \lambda \to \infty$

  Were N(λ) is the density of frequencies
- Use an Abelian theorem:

$$\frac{\lambda}{2\pi N(\lambda)}\sum_{n=1}^{\infty} e^{-\lambda_n t} = \frac{1}{|\Omega|}\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{1}{2\pi t}$$

Physically interesting related problem

- Solution to Shrödinger equations are solutions to the same eigenvalue problem
- Probability of finding a particle in Ω at r:

$$\frac{\sum_{n=1}^{\infty} \exp(-\lambda_n \tau)\, \psi_n^2(\vec{r})}{Z} dr \quad , \tau = \frac{\hbar^2}{m k T}$$

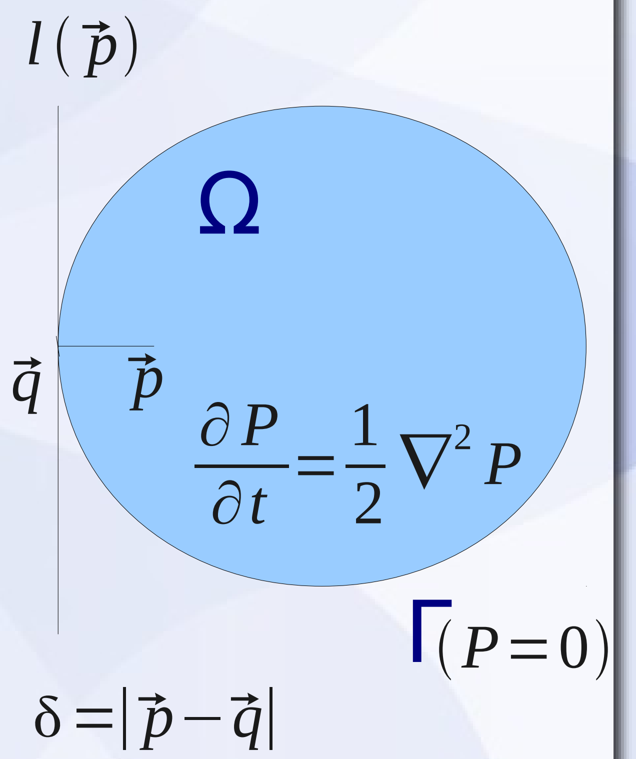
for small τ results must agree with classical statistical mechanics:

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \psi_n^2(r) \sim \frac{1}{|\Omega|} \sum_{n=1}^{\infty} e^{-\lambda_n \tau} \quad , \tau \rightarrow 0$$

Second term

- Consider solutions to diffusion equation on Ω starting from a Dirac delta at point p: $P_{\Omega}(\vec{\rho},\vec{r};t)$
- Consider solution to diffusion equation with the boundary Γ replaced by line l: $P_{l(\vec{p})}(\vec{\rho},\vec{r};t)$
- Note for small t:

$$\int_{\Omega} P_{\Omega}(\vec{\rho},\vec{\rho};t)\,d\vec{\rho} \sim \int_{\Omega} P_{l(\vec{p})}(\vec{\rho},\vec{\rho};t)\,d\vec{\rho}$$



Second Term continued

- Solve diffusion equation in simpler geometry to obtain: $P_{l(\vec{r})}(\vec{\rho},\vec{\rho};t)=\frac{1-e^{-2\delta^2/t}}{2\pi t}(\delta=|\vec{p}-\vec{q}|)$

$$\int_{\Omega} P_{\Omega}(\vec{\rho},\vec{\rho};t)\,d\vec{\rho}\sim\frac{|\Omega|}{2\pi t}-\frac{1}{2\pi t}\int_{\Omega} e^{-2\delta^2/t}\,d\vec{\rho}$$

- This integral can be approximated by noting that as t becomes very small, only contributions from small δs matter:

$$\int_{\Omega} e^{-2\delta^2/t}\,d\vec{\rho}\sim L\int_{0}^{\delta_0} e^{-2\delta^2/t}\sim\frac{L\sqrt{2\pi t}}{4}$$

- Where L is the length of the boundary

Final result for second term

- Putting everything together we now have:

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4\sqrt{2\pi t}}$$

- Consider: $L^2 \geq 4\pi|\Omega|$

  the equality only holds for a circle therefore an important result:

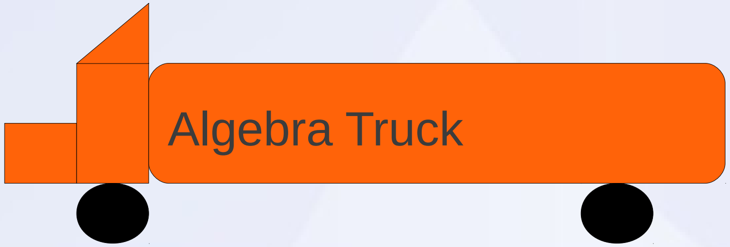
<u>No other shape of drum can sound like a circular drum</u>

Physical Intuition

- First 2 terms should not be surprising:
  - In high frequency limit: wave effects are not noticeable except for very near the boundaries because wavelength is much smaller than any other scale of the system, think of the light in this room
  - Second term is a correction term because of wave effects near the boundaries, but the wavelength is still much smaller than the radius of curvature (smooth boundaries) therefore can only be proportional to length

Third Term

- Consider polygonal drums
- solve with initial delta function for a single wedge to get a more accurate $\int_{\Omega} P_{\Omega}(\vec{\rho}, \vec{\rho}; t)\, d\vec{\rho}$
- Take limit for infinite number of wedges by making wedge angle go to π, turns polygon into smooth curve
- Unfortunately involves (relatively) large amount of algebra not very conducive for slide shows



Polygonal drum

- For a simply connected polygonal drum:

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4\sqrt{2\pi t}}$$

$$+\left(\frac{1}{8\pi}\right)\sum_{\theta} \sin\left(\frac{\pi^2}{\theta}\right)\int_{-\infty}^{\infty}\left(\frac{dy}{(1+\cosh(y))(\cos(\frac{\pi}{\theta}y)-\cos(\frac{\pi^2}{\theta}))}\right)$$

- Taking θ goes to π for an infinite number of angles makes the constant term approach:

$$\left(\frac{1}{4}\right)\int_{-\infty}^{\infty}\left(\frac{dy}{(1+\cosh(y))^2}\right)=\frac{1}{6}$$

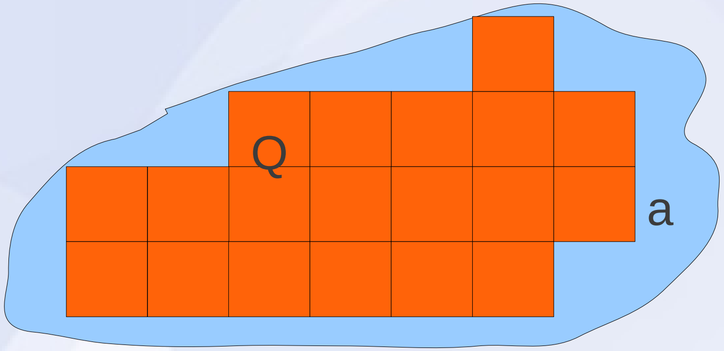
Multiply connected drums

- Letting the polygons approach smooth curves for the exterior angles of a polygon gives -1/6, therefore for every smooth hole 1/6 must be subtracted from the constant part yielding (for a drum with r smooth holes):

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4\sqrt{2\pi t}} + \frac{1}{6}(1-r)$$

First Term (sketch of approach 2)

- More rigorous approach
- Consider diffusion equation with Ω divided into squares of length a, not counting squares which are not entirely inside of Ω



$$P_Q(\vec{\rho},\vec{\rho};t)=\frac{4}{a^2}\sum_{m,n\,odd}\exp\left(\frac{-(m^2+n^2)\pi^2}{2a^2}t\right)\leq P_\Omega(\vec{\rho},\vec{\rho};t)$$

First term second approach cont.

- Also consider solution with no matter destroyed on the boundary, this gives

$$P_\Omega(\vec{\rho},\vec{r};t) \leq \frac{\exp\left(\frac{-|\vec{\rho}-\vec{r}|^2}{2t}\right)}{2\pi t}$$

- Therefore

$$\frac{4}{a^2}\sum_{m,n\ odd}\exp\left(\frac{-(m^2+n^2)\pi^2}{2a^2}t\right) \leq P_\Omega(\vec{\rho},\vec{\rho};t) \leq \frac{1}{2\pi t}$$

  become equalities as a goes to zero and t goes to zero, integrating ρ over Ω gives same first term as other method

References

- Can One Hear the Shape of a Drum?,

  Mark Kac, American Mathematical Monthly, Vol. 3, No. 4, Part 2 Papers in Analysis (Apr. 1966), pp. 1-23

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